

# Semi-equivelar and vertex-transitive maps on the torus

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## Abstract

A vertex-transitive map  $X$  is a map on a closed surface on which the automorphism group  $\text{Aut}(X)$  acts transitively on the set of vertices. If the face-cycles at all the vertices in a map are of same type then the map is said to be a semi-equivelar map. Clearly, a vertex-transitive map is semi-equivelar. Converse of this is not true in general. We show that there are eleven types of semi-equivelar maps on the torus. Three of these are equivelar maps. It is known that two of the three types of equivelar maps on the torus are always vertex-transitive. We show that this is true for the remaining one type of equivelar map and one other type of semi-equivelar maps, namely, if  $X$  is a semi-equivelar map of type  $[6^3]$  or  $[3^3, 4^2]$  then  $X$  is vertex-transitive. We also show, by presenting examples, that this result is not true for the remaining seven types of semi-equivelar maps. There are ten types of semi-equivelar maps on the Klein bottle. We present examples in each of the ten types which are not vertex-transitive.

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## 1 Introduction

By a map we mean a polyhedral map on a surface. So, a face of a map is a  $p$ -gon for some integer  $p \geq 3$ . A map  $X$  is said to be *weakly regular* or *vertex-transitive* if the automorphism group  $\text{Aut}(X)$  acts transitively on the set  $V(X)$  of vertices of  $X$ .

If  $v$  is a vertex in a map  $X$  then the faces containing  $v$  form a cycle (called the *face-cycle*)  $C_v$  in the dual graph  $\Lambda(X)$  of  $X$ . Clearly,  $C_v$  is of the form  $P_1-P_2-\cdots-P_k-P_1$ , where  $P_i$  is a path consisting of  $n_i$   $p_i$ -gons and  $p_i \neq p_{i+1}$  for  $1 \leq i \leq k$  (addition in the suffix is modulo  $k$ ). A map  $X$  is called *semi-equivelar* (or *semi-regular* if  $C_u$  and  $C_v$  are of same type for any two vertices  $u$  and  $v$  of  $X$ ). More precisely, there exist natural numbers  $p_1, \dots, p_k \geq 3$  and  $n_1, \dots, n_k \geq 1$ ,  $p_i \neq p_{i+1}$  such that both  $C_u$  and  $C_v$  are of the form  $P_1-P_2-\cdots-P_k-P_1$  as above, where  $P_i$  is a path consists of  $n_i$   $p_i$ -gons. In this case, we say that  $X$  is *semi-equivelar of type*  $[p_1^{n_1}, \dots, p_k^{n_k}]$ . (We identify  $[p_1^{n_1}, \dots, p_k^{n_k}]$  with  $[p_2^{n_2}, \dots, p_k^{n_k}, p_1^{n_1}]$  and with  $[p_k^{n_k}, \dots, p_1^{n_1}]$ .) An *equivelar* map (of type  $[p^q]$ ,  $(p, q)$  or  $\{p, q\}$ ) is a semi-equivelar map (of type  $[p^q]$  for some  $p, q \geq 3$ ). Clearly, a vertex-transitive map is semi-equivelar.

A *semi-regular* tiling of the plane  $\mathbb{R}^2$  is a tiling of  $\mathbb{R}^2$  by regular polygons such that all the vertices of the tiling are of same type. A *semi-regular* tiling of  $\mathbb{R}^2$  is also known as *Archimedean*, or *homogeneous*, or *uniform* tiling. In [7], Grünbaum and Shephard showed

that there are exactly eleven types of Archimedean tilings on the plane. These types are  $[3^6]$ ,  $[3^4, 6^1]$ ,  $[3^3, 4^2]$ ,  $[3^2, 4^1, 3^1, 4^1]$ ,  $[3^1, 6^1, 3^1, 6^1]$ ,  $[3^1, 4^1, 6^1, 4^1]$ ,  $[3^1, 12^2]$ ,  $[4^4]$ ,  $[4^1, 6^1, 12^1]$ ,  $[4^1, 8^2]$ ,  $[6^3]$ . Clearly, a *semi-regular* tiling on  $\mathbb{R}^2$  gives a semi-equivelar map on  $\mathbb{R}^2$ . But, there are semi-equivelar maps on the plane which are not (not isomorphic to) an Archimedean tiling. In fact, there exists  $[p^q]$  equivelar maps on  $\mathbb{R}^2$  whenever  $1/p + 1/q < 1/2$  (e.g.,  $[2]$ ,  $[6]$ ). Thus, we have

**Proposition 1.1.** *There are infinitely many types of equivelar maps on the plane  $\mathbb{R}^2$ .*

All vertex-transitive maps on the 2-sphere are known. These are the boundaries of Platonic and Archimedean solids and two infinite families of types (namely, of types  $[4^2, n^1]$  and  $[3^3, m^1]$  for  $4 \neq n \geq 3$ ,  $m \geq 4$ ) [7]. Similarly, there are infinitely many types of vertex-transitive maps on the real projective plane [1]. Thus, there are infinitely many types of semi-equivelar maps on the 2-sphere and the real projective plane. But, for a surface of negative Euler characteristic the picture is different. In [1], Babai has shown the following.

**Proposition 1.2.** *A semi-equivelar map on a surface of Euler characteristic  $\chi < 0$  has at most  $-84\chi$  vertices.*

As a consequence of this we get

**Corollary 1.3.** *If the Euler characteristic  $\chi(M)$  of a surface  $M$  is negative then the number of semi-equivelar maps on  $M$  is finite.*

We know from [4] and [5] that infinitely many equivelar maps exist on both the torus and the Klein bottle. Thus, infinitely many semi-equivelar maps exist on both the torus and the Klein bottle. But, only eleven types of semi-equivelar maps on the torus and ten types of semi-equivelar maps on the Klein bottle are known. All these are quotients of Archimedean tilings of the plane [1, 11, 12]. Since there are infinitely many equivelar maps exist on the plane, it is natural to ask whether there are more types of semi-equivelar maps exist on the torus or the Klein bottle. Here we prove

**Theorem 1.4.** *Let  $X$  be a semi-equivelar map on a surface  $M$ . If  $M$  is the torus or the Klein bottle then the type of  $X$  is  $[3^6]$ ,  $[6^3]$ ,  $[4^4]$ ,  $[3^4, 6^1]$ ,  $[3^3, 4^2]$ ,  $[3^2, 4^1, 3^1, 4^1]$ ,  $[3^1, 6^1, 3^1, 6^1]$ ,  $[3^1, 4^1, 6^1, 4^1]$ ,  $[3^1, 12^2]$ ,  $[4^1, 8^2]$  or  $[4^1, 6^1, 12^1]$ .*

It was shown in [5] that every equivelar map of type  $[3^6]$  on the torus is vertex-transitive. By the similar arguments, one can easily show that a equivelar map of type  $[4^4]$  on the torus is vertex-transitive. Thus, we have

**Proposition 1.5.** *Let  $X$  be an equivelar map on the torus. If the type of  $X$  is  $[3^6]$  or  $[4^4]$  then  $X$  is vertex-transitive.*

Here we prove

**Theorem 1.6.** *Let  $X$  be a semi-equivelar map on the torus. If the type of  $X$  is  $[6^3]$  or  $[3^3, 4^2]$  then  $X$  is vertex-transitive.*

In Section 4, we present examples of the other seven types of semi-equivelar maps which are not vertex-transitive. This proves

**Theorem 1.7.** *If  $[p_1^{n_1}, \dots, p_k^{n_k}] = [3^2, 4^1, 3^1, 4^1]$ ,  $[3^4, 6^1]$ ,  $[3^1, 6^1, 3^1, 6^1]$ ,  $[3^1, 4^1, 6^1, 4^1]$ ,  $[3^1, 12^2]$ ,  $[4^1, 8^2]$  or  $[4^1, 6^1, 12^1]$  then there exists a semi-equivelar map of type  $[p_1^{n_1}, \dots, p_k^{n_k}]$  on the torus which is not vertex-transitive.*

In [9], the second author and Upadhyay have shown that there does not exist any semi-equivelar map of type  $[3^4, 6^1]$  on the Klein bottle. In [5], the first author and Upadhyay have presented examples of  $[3^6]$  equivelar maps on the Klein bottle which are not vertex-transitive. In Section 4, we present examples of the other nine types of semi-equivelar maps on the Klein bottle which are not vertex-transitive. Thus, we have

**Theorem 1.8.** *If  $[p_1^{n_1}, \dots, p_k^{n_k}]$  is one in the list of 11 types in Theorem 1.4 and  $[p_1^{n_1}, \dots, p_k^{n_k}] \neq [3^4, 6^1]$  then there exists a semi-equivelar map of type  $[p_1^{n_1}, \dots, p_k^{n_k}]$  on the Klein bottle which is not vertex-transitive.*

Several examples of  $[3^6]$  and  $[4^4]$  equivelar maps on the torus are in [4]. From this, one can construct equivelar maps of type  $[6^3]$  on the torus. In Example 4.1, we also present an equivelar map of type  $[3^3, 4^2]$  on the torus for the sake of completeness.

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## 2 Proofs of Theorem 1.4 and Proposition 1.1

For  $n \geq 3$ , the  $n$ -gon whose edges are  $u_1u_2, \dots, u_{n-1}u_n, u_nu_1$  is denoted by  $u_1u_2 \cdots u_nu_1$  or by  $C_n(u_1, \dots, u_n)$ . We call 3-gons and 4-gons by *triangles* and *quadrangles* respectively. A triangle  $u-v-w-u$  is also denoted by  $uvw$ . If  $X$  is a map on a surface  $M$  then we identify a face of  $X$  in  $M$  with the boundary cycle of the face.

*Proof of Proposition 1.1.* In [4], it was shown that there exists equivelar map of type  $[3^8]$  on the orientable surface of genus  $g$  for each  $g \geq 4$ . For a fixed  $g \geq 4$ , let  $X$  be one such equivelar map of type  $[3^8]$  on the surface  $M_g$  of genus  $g$ . Since the 2-disk  $\mathbb{D}^2$  is the universal cover of  $M_g$ , by pulling back  $X$ , we get an equivelar map  $\tilde{X}$  of type  $[3^8]$  on  $\mathbb{D}^2$  and a polyhedral map  $\eta : \tilde{X} \rightarrow X$ . From the constructions in [3] and [4], we know that an equivelar map of type  $[p^q]$  exists on some surface (orientable or non-orientable) of appropriate genus for each  $[p^q]$  in  $\{[3^7], [4^5], [4^6], [3^{3\ell-1}], [3^{3\ell}], [k^k] : \ell \geq 3, k \geq 5\}$ . So, by the same arguments, equivelar maps of types  $[p^q]$  exist on  $\mathbb{D}^2$  for  $[p^q]$  in  $\{[3^7], [4^5], [4^6], [3^{3\ell-1}], [3^{3\ell}], [k^k] : \ell \geq 3, k \geq 5\}$ . More generally, there exist equivelar maps of type  $[p^q]$  on  $\mathbb{D}^2$  whenever  $1/p + 1/q < 1/2$  (cf., [2], [6], [7]). Since  $\mathbb{R}^2$  is homeomorphic to  $\mathbb{D}^2$ , an equivelar map of type  $[p^q]$  determines an equivelar map of type  $[p^q]$  on  $\mathbb{R}^2$ . Thus, there exist equivelar maps of types  $[p^q]$  on  $\mathbb{R}^2$  whenever  $1/p + 1/q < 1/2$ . The result now follows.  $\square$

**Lemma 2.1.** *Let  $X$  be a semi-equivelar map on a surface  $M$ . If  $\chi(M) = 0$  then the type of  $X$  is  $[3^6]$ ,  $[3^4, 6^1]$ ,  $[3^3, 4^2]$ ,  $[3^2, 4^1, 3^1, 4^1]$ ,  $[4^4]$ ,  $[3^1, 6^1, 3^1, 6^1]$ ,  $[3^2, 6^2]$ ,  $[3^2, 4^1, 12^1]$ ,  $[3^1, 4^1, 3^1, 12^1]$ ,  $[3^1, 4^1, 6^1, 4^1]$ ,  $[3^1, 4^2, 6^1]$ ,  $[6^3]$ ,  $[3^1, 12^2]$ ,  $[4^1, 8^2]$ ,  $[5^2, 10^1]$ ,  $[3^1, 7^1, 42^1]$ ,  $[3^1, 8^1, 24^1]$ ,  $[3^1, 9^1, 18^1]$ ,  $[3^1, 10^1, 15^1]$ ,  $[4^1, 5^1, 20^1]$  or  $[4^1, 6^1, 12^1]$ .*

*Proof.* Let the type of  $X$  be  $[p_1^{n_1}, \dots, p_k^{n_k}]$ . Consider the  $\ell$ -tuple  $(q_1^{m_1}, \dots, q_\ell^{m_\ell})$ , where  $q_i \neq q_j$  for  $i \neq j$ ,  $q_i = p_j$  for some  $j$ ,  $m_i = \sum_{p_i=q_j} n_j$  and  $(m_1, q_1) > (m_2, q_2) > \dots > (m_\ell, q_\ell)$ . (Here,  $(m, p) > (n, q)$  means either (i)  $m > n$  or (ii)  $m = n$  and  $p < q$ .)

*Claim.*  $(q_1^{m_1}, \dots, q_\ell^{m_\ell}) = (3^6)$ ,  $(3^4, 6^1)$ ,  $(3^3, 4^2)$ ,  $(4^4)$ ,  $(3^2, 6^2)$ ,  $(3^2, 4^1, 12^1)$ ,  $(4^2, 3^1, 6^1)$ ,  $(6^3)$ ,  $(12^2, 3^1)$ ,  $(8^2, 4^1)$ ,  $(5^2, 10^1)$ ,  $(3^1, 7^1, 42^1)$ ,  $(3^1, 8^1, 24^1)$ ,  $(3^1, 9^1, 18^1)$ ,  $(3^1, 10^1, 15^1)$ ,  $(4^1, 5^1, 20^1)$  or  $(4^1, 6^1, 12^1)$ .

Let  $f_0, f_1$  and  $f_2$  denote the number of vertices, edges and faces of  $X$  respectively. Let  $d$  be the degree of each vertex. Then,  $d = n_1 + \dots + n_k = m_1 + \dots + m_\ell$  and  $f_1 = f_0 \times d/2$ . Clearly, the number of  $q_i$ -gons is  $f_0 \times m_i/q_i$ . This implies that  $f_2 = f_0(m_1/q_1 + \dots + m_\ell/q_\ell)$ . Since  $\chi(M) = 0$ , it follows that  $f_0 - f_0(m_1 + \dots + m_\ell)/2 + f_0(m_1/q_1 + \dots + m_\ell/q_\ell) = 0$  or

$$\left(\frac{1}{2} - \frac{1}{q_1}\right)m_1 + \dots + \left(\frac{1}{2} - \frac{1}{q_\ell}\right)m_\ell = 1. \quad (1)$$

Since  $q_i \geq 3$ , it follows that  $d \leq 6$ . Moreover, if  $d = 6$  then  $\ell = 1$  and  $q_1 = 3$ . In this case,  $(q_1^{m_1}, \dots, q_\ell^{m_\ell}) = (3^6)$ .

Now, assume  $d = 5$ . Then  $(m_1, \dots, m_\ell) = (5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1)$  or  $(1, 1, 1, 1, 1)$ . It is easy to see that for  $(m_1, \dots, m_\ell) = (5), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1)$  or  $(1, 1, 1, 1, 1)$ , Equation (1) has no solution. So,  $(m_1, \dots, m_\ell) = (4, 1)$  or  $(3, 2)$ . In the first case,  $(q_1, q_2) = (3, 6)$  and in the second case,  $(q_1, q_2) = (3, 4)$ . Thus,  $(q_1^{m_1}, \dots, q_\ell^{m_\ell}) = (3^4, 6^1)$  or  $(3^3, 4^2)$ .

Let  $d = 4$ . Then  $(m_1, \dots, m_\ell) = (4), (3, 1), (2, 2), (2, 1, 1)$  or  $(1, 1, 1, 1)$ . Again, for  $(m_1, \dots, m_\ell) = (3, 1)$  or  $(1, 1, 1, 1)$ , Equation (1) has no solution. So,  $(m_1, \dots, m_\ell) = (4), (2, 2)$  or  $(2, 1, 1)$ . In the first case,  $q_1 = 4$ , in the second case,  $(q_1, q_2) = (3, 6)$  and in the third case,  $(q_1, \{q_2, q_3\}) = (3, \{4, 12\})$  or  $(4, \{3, 6\})$ . Thus,  $(q_1^{m_1}, \dots, q_\ell^{m_\ell}) = (4^4), (3^2, 6^2), (3^2, 4^1, 12^1)$  or  $(4^2, 3^1, 6^1)$ .

Finally, assume  $d = 3$ . Then  $(m_1, \dots, m_\ell) = (3), (2, 1)$  or  $(1, 1, 1)$ . In the first case,  $q_1 = 6$ , in the second case,  $(q_1, q_2) = (12, 3), (8, 4)$  or  $(5, 10)$  and in the third case,  $\{q_1, q_2, q_3\} = \{3, 7, 42\}, \{3, 8, 24\}, \{3, 9, 18\}, \{3, 10, 15\}, \{4, 5, 20\}$  or  $\{4, 6, 12\}$ . Thus,  $(q_1^{m_1}, \dots, q_\ell^{m_\ell}) = (6^3), (12^2, 3^1), (8^2, 4^1), (5^2, 10^1), (3^1, 7^1, 42^1), (3^1, 8^1, 24^1), (3^1, 9^1, 18^1), (3^1, 10^1, 15^1), (4^1, 5^1, 20^1)$  or  $(4^1, 6^1, 12^1)$ . This proves the claim.

The lemma follows from the claim.  $\square$

We need the following technical lemma for the proof of Theorem 1.4.

**Lemma 2.2.** *If  $[p_1^{n_1}, \dots, p_k^{n_k}]$  satisfies any of the following three properties then  $[p_1^{n_1}, \dots, p_k^{n_k}]$  can not be the type of any semi-equivelar map on a surface.*

- (i) *There exists  $i$  such that  $n_i = 2$ ,  $p_i$  is odd and  $p_j \neq p_i$  for all  $j \neq i$ .*
- (ii) *There exists  $i$  such that  $n_i = 1$ ,  $p_j \neq p_i$  for all  $j \neq i$  and  $p_{i-1} \neq p_{i+1}$ .*
- (iii) *There exists  $i$  such that  $n_i = 1$ ,  $p_i$  is odd,  $p_{i-1} \neq p_j$  for all  $j \neq i-1$  and  $p_{i+1} \neq p_\ell$  for all  $\ell \neq i+1$ .*

(Here, addition in the subscripts are modulo  $k$ .)

*Proof.* If possible let there exist a semi-equivelar map  $X$  of type  $[p_1^{n_1}, \dots, p_k^{n_k}]$  which satisfies

(i). Let  $A = u_1 - u_2 - u_3 - \dots - u_{p_i} - u_1$  be a  $p_i$ -gon. Let the other face containing  $u_r - u_{r+1}$  be  $A_r$  for  $1 \leq r \leq p_i$ . (Addition in the subscripts are modulo  $p_i$ .) Consider the face-cycle of the vertex  $u_1$ . Since  $p_j \neq p_i$  for all  $j \neq i$  and  $n_i = 2$ , it follows that exactly one of  $A_1$  and  $A_{p_i}$  is a  $p_i$ -gon. Assume, without loss, that  $A_1$  is a  $p_i$ -gon. Since  $u_2$  is in two  $p_i$ -gons, it follows that  $A_2$  is not a  $p_i$ -gon. Therefore (by considering the vertex  $u_3$ , as in the case for the vertex  $u_1$ ),  $A_3$  is a  $p_i$ -gon. Continuing this way, we get  $A_1, A_3, A_5, \dots$  are  $p_i$ -gons. Since  $p_i$  is odd, it follows that  $A_{p_i}$  is a  $p_i$ -gon. Then we get three  $p_i$ -gons, namely,  $A$ ,  $A_1$  and  $A_{p_i}$ , through  $u_1$ . This is a contradiction.

If possible let there exist a semi-equivelar map  $Y$  of type  $[p_1^{n_1}, \dots, p_k^{n_k}]$  which satisfies (ii). Let  $B = u_1-u_2-u_3-\dots-u_{p_i}-u_1$  be a  $p_i$ -gon. Let the other face containing  $u_r u_{r+1}$  be  $B_r$  for  $1 \leq r \leq p_i$ . Consider the face-cycle of the vertex  $u_2$ . Since  $p_j \neq p_i$  and  $n_i = 1$ ,  $A$  is the only  $p_i$ -gon containing  $u_2$ . Since  $p_{i-1} \neq p_{i+1}$ , it follows that one of  $B_1$  and  $B_2$  is a  $p_{i-1}$ -gon and the other is a  $p_{i+1}$ -gon. Assume, without loss, that  $B_1$  is a  $p_{i-1}$ -gon and  $B_2$  is a  $p_{i+1}$ -gon. Then, by the same argument as above,  $B_1, B_3, B_5, \dots$  are  $p_{i-1}$ -gons and  $B_2, B_4, \dots$  are  $p_{i+1}$ -gons. Since  $p_i$  is odd, it follows that  $B_{p_i}$  is a  $p_{i-1}$ -gon. Then, from the face-cycle of  $u_1$ , it follows that  $p_{i+1} = p_{i-1}$ . This contradicts the assumption.

Finally, assume that there exists a semi-equivelar map  $Z$  of type  $[p_1^{n_1}, \dots, p_k^{n_k}]$  which satisfies (iii). Let  $P$  and  $Q$  be two adjacent faces through a vertex  $u_1$ , where  $P$  is a  $p_i$ -gon and  $Q$  is a  $p_{i-1}$ -gon. Assume that  $P = u_1-u_2-u_3-\dots-u_{p_i}-u_1$  and  $Q = u_1-v_2-v_3-\dots-v_{p_{i-1}-1}-u_{p_i}-u_1$ . Let the other face containing  $u_r u_{r+1}$  be  $P_r$  for  $1 \leq r \leq p_i$ . (Addition in the subscripts are modulo  $p_i$ .) Since  $p_{i-1} \neq p_j$  for all  $j \neq i-1$  and  $p_{i+1} \neq p_\ell$  for all  $\ell \neq i+1$ , considering the face-cycle of  $u_1$ , it follows that  $P_1$  is a  $p_{i+1}$ -gon. Considering the face-cycle of  $u_2$ , by the similar argument (interchanging  $p_{i-1}$  and  $p_{i+1}$ ), it follows that  $P_2$  is a  $p_{i-1}$ -gon. Continuing this way, we get  $P_1, P_3, \dots$  are  $p_{i+1}$ -gons and  $P_2, P_4, \dots$  are  $p_{i-1}$ -gons. Since  $p_i$  is odd, it follows that  $P_{p_i}$  is a  $p_{i+1}$ -gon. This is a contradiction since  $P_{p_i} = Q$  is a  $p_{i-1}$ -gon and  $p_{i-1} \neq p_{i+1}$ . This completes the proof.  $\square$

*Proof of Theorem 1.4.* Let  $X$  be a semi-equivelar map of type  $[p_1^{n_1}, \dots, p_k^{n_k}]$  on the torus or the Klein bottle. Then, by Lemma 2.2 (i),  $[p_1^{n_1}, \dots, p_k^{n_k}] \neq [3^2, 6^2], [3^2, 4^1, 12^1], [5^2, 10^1]$ . Again, by Lemma 2.2 (ii),  $[p_1^{n_1}, \dots, p_k^{n_k}] \neq [3^1, 4^2, 6^1], [3^1, 7^1, 42^1], [3^1, 8^1, 24^1], [3^1, 9^1, 18^1], [3^1, 10^1, 15^1], [4^1, 5^1, 20^1]$ . Finally, by Lemma 2.2 (iii),  $[p_1^{n_1}, \dots, p_k^{n_k}] \neq [3^1, 4^1, 3^1, 12^1]$ . The result now follows by Lemma 2.1.  $\square$

### 3 Proof of Theorem 1.6

A triangulation of a 2-manifold is called *degree-regular* if each of its vertices have the same degree. In other word, a degree-regular triangulation is an equivelar map of type  $[3^k]$  for some  $k \geq 3$ . The triangulation  $E_1$  given below is a degree-regular triangulation of  $\mathbb{R}^2$ .

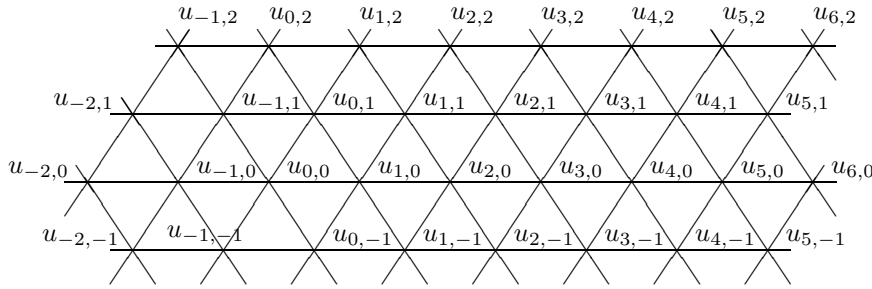


Figure 1: Regular  $[3^6]$ -tiling  $E_1$  of  $\mathbb{R}^2$

From [5] we know

**Proposition 3.1.** *Let  $M$  be a triangulation of the plane  $\mathbb{R}^2$ . If the degree of each vertex of  $M$  is 6 then  $M$  is isomorphic to  $E_1$ .*

Using Proposition 3.1, it was shown in [5] that ‘any degree-regular triangulation of the torus is vertex-transitive’. Here we prove

**Lemma 3.2.** *Let  $X$  be a triangulation of the torus. If  $X$  is degree-regular then the automorphism group  $\text{Aut}(X)$  acts face-transitively on  $X$ .*

*Proof.* Since  $X$  is degree-regular and Euler characteristic of  $X$  is 0, it follows that the degree of each vertex in  $X$  is 6.

Since  $\mathbb{R}^2$  is the universal cover of the torus, there exists a triangulation  $Y$  of  $\mathbb{R}^2$  and a simplicial covering map  $\eta: Y \rightarrow X$  (cf. [10, Page 144]). Since the degree of each vertex in  $X$  is 6, the degree of each vertex in  $Y$  is 6. Because of Proposition 3.1, we may assume that  $Y = E_1$ . Let  $\Gamma_1$  be the group of covering transformations. Then  $|X| = |E_1|/\Gamma_1$ .

We take  $V_1 = \{u_{i,2j} = (i, j\sqrt{3}), u_{i,2j+1} = (i + 1/2, (2j + 1)\sqrt{3}/2) : i, j \in \mathbb{Z}\}$  as the vertex set of  $E_1$ . Then  $H_1 := \{x \mapsto x + a, a \in V_1\}$  is a subgroup of  $\text{Aut}(E_1)$  and is called the group of translations. Clearly,  $H_1$  is commutative.

For  $\sigma \in \Gamma_1$ ,  $\eta \circ \sigma = \eta$ . So,  $\sigma$  maps the geometric carrier of a simplex to the geometric carrier of a simplex. This implies that  $\sigma$  induces an automorphism  $\sigma$  of  $E_1$ . Thus, we can identify  $\Gamma_1$  with a subgroup of  $\text{Aut}(E_1)$ . So,  $X$  is a quotient of  $E_1$  by the subgroup  $\Gamma_1$  of  $\text{Aut}(E_1)$ , where  $\Gamma_1$  has no fixed element (vertex, edge or face). Hence  $\Gamma_1$  consists of translations and glide reflections. Since  $X = E_1/\Gamma_1$  is orientable,  $\Gamma_1$  does not contain any glide reflection. Thus  $\Gamma_1 \leq H_1$ .

Consider the subgroup  $G_1$  of  $\text{Aut}(E_1)$  generated by  $H_1$  and the map  $x \mapsto -x$ . So,

$$G_1 = \{\alpha : x \mapsto \varepsilon x + a : \varepsilon = \pm 1, a \in V_1\} \cong H_1 \rtimes \mathbb{Z}_2.$$

*Claim 1.*  $G_1$  acts face-transitively on  $E_1$ .

Since  $H_1$  is vertex transitively on  $E_1$ , to prove Claim 1, it is sufficient to show that  $G_1$  acts transitively on the set of six faces containing  $u_{0,0}$ . This follows from the following:  $u_{-1,0}u_{0,0}u_{-1,1} + u_{1,0} = u_{0,0}u_{1,0}u_{0,1} = u_{-1,-1}u_{0,-1}u_{0,0} + u_{0,1}$ ,  $u_{-1,0}u_{-1,-1}u_{0,0} + u_{1,0} = u_{0,0}u_{0,-1}u_{1,0} = u_{-1,1}u_{0,0}u_{0,1} + u_{0,-1}$  and  $-1 \cdot u_{0,0}u_{-1,0}u_{-1,-1} = u_{0,0}u_{1,0}u_{0,1}$ .

*Claim 2.* If  $K \leq H_1$  then  $K \trianglelefteq G_1$ .

Let  $\alpha \in G_1$  and  $\beta \in K$ . Assume  $\alpha(x) = \varepsilon x + a$  and  $\beta(x) = x + b$  for some  $a, b \in V(E_1)$  and  $\varepsilon \in \{1, -1\}$ . Then  $(\alpha \circ \beta \circ \alpha^{-1})(x) = (\alpha \circ \beta)(\varepsilon(x - a)) = \alpha(\varepsilon(x - a) + b) = x - a + \varepsilon b + a = x + \varepsilon b = \beta^\varepsilon(x)$ . Thus,  $\alpha \circ \beta \circ \alpha^{-1} = \beta^\varepsilon \in K$ . This proves Claim 2.

By Claim 2,  $\Gamma_1 \trianglelefteq G_1$  and hence we can assume that  $G_1/\Gamma_1 \leq \text{Aut}(E_1/\Gamma_1)$ . Since, by Claim 1,  $G_1$  acts face-transitively on  $E_1$ , it follows that  $G_1/\Gamma_1$  acts face-transitively on  $E_1/\Gamma_1$ . This completes the proof since  $X = E_1/\Gamma_1$ .  $\square$

We need the following two lemmas for the proof of Theorem 1.6.

**Lemma 3.3.** *Let  $X$  be a map on the 2-disk  $\mathbb{D}^2$  whose faces are triangles and quadrangles. For a vertex  $x$  of  $X$ , let  $n_3(x)$  and  $n_4(x)$  be the number of triangles and quadrangles through  $x$  respectively. Suppose  $(n_3(u), n_4(u)) = (3, 2)$  for each internal vertex  $u$ . Then  $X$  does not satisfy any of the following.*

- (a)  $1 \leq n_4(w) \leq 2$ ,  $n_3(w) + n_4(w) \leq 4$  for one vertex  $w$  on the boundary, and  $(n_3(v), n_4(v)) = (0, 2)$  for each boundary vertex  $v \neq w$ .
- (b)  $1 \leq n_3(w) \leq 3$ ,  $n_4(w) \leq 2$  and  $n_3(w) + n_4(w) \leq 4$  for one vertex  $w$  on the boundary, and  $(n_3(v), n_4(v)) = (3, 0)$  for each boundary vertex  $v \neq w$ .

*Proof.* Let  $f_0, f_1$  and  $f_2$  denote the number of vertices, edges and faces of  $X$  respectively. Let  $n_3$  (resp.,  $n_4$ ) denote the total number of triangles (resp., quadrangles) in  $X$ . Let there be  $n$  internal vertices and  $m + 1$  boundary vertices. So,  $f_0 = n + m + 1$  and  $f_2 = n_3 + n_4$ .

If possible let  $X$  satisfy (a). Then  $n_4 = (2n + 2m + n_4(w))/4$  and  $n_3 = (3n + n_3(w))/3$ . Since  $1 \leq n_4(w) \leq 2$ , it follows that  $n_4(w) = 2$  and hence  $n_3(w) \leq 2$ . These imply that  $n_3(w) = 0$ . Thus, the exceptional vertex is like other boundary vertices. Therefore, each boundary vertex is in three edges and hence  $f_1 = (5n + 3m + 3)/2$ . These imply  $f_0 - f_1 + f_2 = (n + m + 1) - (5n + 3m + 3)/2 + (n + (n + m + 1)/2) = 0$ . This is not possible since the Euler characteristic of the 2-disk  $\mathbb{D}^2$  is 1.

If  $X$  satisfies (b) then  $n_3 = (3n + 3m + n_3(w))/3$  and  $n_4 = (2n + n_4(w))/4$ . Since  $1 \leq n_3(w) \leq 3$ , it follows that  $n_3(w) = 3$  and hence  $n_4(w) \leq 1$ . These imply that  $n_4(w) = 0$ . Thus, the exceptional vertex is like other boundary vertices and each boundary vertex is in four edges. Thus,  $f_1 = (5n + 4m + 4)/2$  and  $f_2 = n_4 + n_3 = 3n/2 + m + 1$ . Then  $f_0 - f_1 + f_2 = (n + m + 1) - (5n + 4m + 4)/2 + (3n/2 + m + 1) = 0$ , a contradiction again. This completes the proof.  $\square$

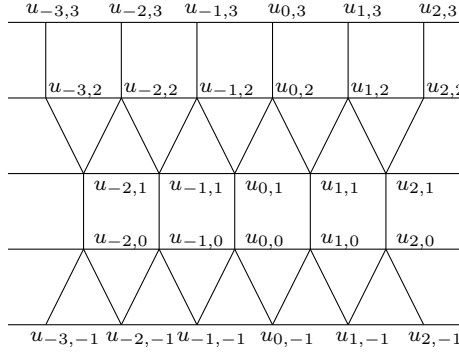


Figure 2: Elongated triangular tiling  $E_2$ ,

**Lemma 3.4.** *Let  $E_2$  be the Archimedean tiling of the plane  $\mathbb{R}^2$  given in Figure 2. If  $X$  is a semi-equivelar map of  $\mathbb{R}^2$  of type  $[3^3, 4^2]$  then  $X \cong E_2$ .*

*Proof.* Let the type of  $X$  be  $[3^3, 4^2]$ . Choose a vertex  $v_{0,0}$ . Let the two quadrangle through  $v_{0,0}$  be  $v_{-1,0}-v_{0,0}-v_{0,1}-v_{-1,1}-v_{-1,0}$  and  $v_{0,0}-v_{1,0}-v_{1,1}-v_{0,1}-v_{0,0}$ . Then the second quadrangle through  $v_{1,0}$  is of the form  $v_{1,0}-v_{2,0}-v_{2,1}-v_{1,1}-v_{1,0}$  and the second quadrangle through  $v_{-1,0}$  is of the form  $v_{-2,0}-v_{-1,0}-v_{-1,1}-v_{-2,1}-v_{-2,0}$ . Continuing this way, we get a path  $P_0 := \cdots -v_{-2,0}-v_{-1,0}-v_{0,0}-v_{1,0}-v_{2,0}-\cdots$  in the edge graph of  $X$  such that all the quadrangles incident with a vertex of  $P_0$  lie on one side of  $P_0$  and all the triangles incident with the same vertex lie on the other side of  $P_0$ . If  $P_0$  has a closed sub-path then  $P_0$  contains a cycle  $W$ . In that case, the bounded part of  $X$  with boundary  $W$  is a map on the 2-disk  $\mathbb{D}^2$  which satisfies (a) or (b) of Lemma 3.3. This is not possible by Lemma 3.3. Thus,  $P_0$  is an infinite path. Then the faces through vertices of  $P_0$  forms an infinite strip which is bounded by two infinite paths, say  $P_{-1} = \cdots -v_{-2,-1}-v_{-1,-1}-v_{0,-1}-v_{1,-1}-v_{2,-1}-\cdots$  and  $P_1 = \cdots -v_{-2,1}-v_{-1,1}-v_{0,1}-v_{1,1}-v_{2,1}-\cdots$ , where the faces between  $P_0$  and  $P_1$  are quadrangles and the faces between  $P_0$  and  $P_{-1}$  are triangles and the faces through  $v_{i,0}$  are  $v_{i-1,0}-v_{i,0}-v_{i,1}-v_{i-1,1}-v_{i-1,0}$ ,  $v_{i,0}-v_{i+1,0}-v_{i+1,1}-v_{i,1}-v_{i,0}$ ,  $v_{i,0}v_{i+1,0}v_{i,-1}$ ,  $v_{i,0}v_{i,-1}v_{i-1,-1}$ ,  $v_{i,0}v_{i-1,-1}v_{i-1,0}$ .

Similarly, starting with the vertex  $v_{0,1}$  in place of  $v_{0,0}$  we get the paths  $P_0, P_1, P_2 = \cdots -v_{-2,2}-v_{-1,2}-v_{0,2}-v_{1,2}-v_{2,2}-\cdots$ , where the faces between  $P_1$  and  $P_2$  are triangles and the triangles through  $u_{i,1}$  are  $v_{i,1}v_{i+1,1}v_{i,2}$ ,  $v_{i,1}v_{i,2}v_{i-1,2}$ ,  $v_{i,1}v_{i-1,2}v_{i-1,1}$ . Continuing this way

we get paths  $\cdots, P_{-2}, P_{-1}, P_0, P_1, P_2, \cdots$  such that the faces between  $P_{2j}$  and  $P_{2j+1}$  are rectangles and the faces between  $P_{2j-1}$  and  $P_{2j}$  are triangles, the five faces through  $v_{i,2j}$  are  $v_{i-1,2j}-v_{i,2j}-v_{i,2j+1}-v_{i-1,2j+1}-v_{i-1,2j}$ ,  $v_{i,2j}-v_{i+1,2j}-v_{i+1,2j+1}-v_{i,2j+1}-v_{i,2j}$ ,  $v_{i,2j}v_{i+1,2j}v_{i,2j-1}$ ,  $v_{i,2j}v_{i-1,2j-1}v_{i-1,2j}$  and the five faces through  $v_{i,2j+1}$  are  $v_{i-1,2j}-v_{i,2j}-v_{i,2j+1}-v_{i-1,2j+1}-v_{i-1,2j}$ ,  $v_{i,2j}-v_{i+1,2j}-v_{i+1,2j+1}-v_{i,2j+1}-v_{i,2j}$ ,  $v_{i,2j+1}v_{i+1,2j+1}v_{i,2j+2}$ ,  $v_{i,2j+1}v_{i,2j+2}v_{i-1,2j+2}$ ,  $v_{i,2j+1}v_{i-1,2j+2}v_{i-1,2j+1}$  for all  $j \in \mathbb{Z}$ . Then the mapping  $f : V(X) \rightarrow V(E_2)$ , given by  $f(v_{k,t}) = u_{k,t}$  for  $k, t \in \mathbb{Z}$ , is an isomorphism. This proves the lemma.  $\square$

*Proof of Theorem 1.6.* Let  $X$  be an equivelar map of type  $[6^3]$  on the torus. Let  $Y$  be the dual of  $X$ . Then  $Y$  is an equivelar map of type  $[3^6]$  on the torus and  $\text{Aut}(Y) \equiv \text{Aut}(X)$ . By Lemma 3.2,  $\text{Aut}(Y)$  acts face-transitively on  $Y$ . These imply,  $\text{Aut}(X)$  acts vertex-transitively on  $X$ . So,  $X$  is vertex-transitive.

Now, assume that  $X$  is a semi-equivelar map of type  $[3^3, 4^2]$  on the torus. Since  $\mathbb{R}^2$  is the universal cover of the torus, by pulling back  $X$  (using similar arguments as in the proof of Theorem 3 in [10, Page 144]), we get a semi-equivelar map  $\tilde{X}$  of type  $[3^3, 4^2]$  on  $\mathbb{R}^2$  and a polyhedral covering map  $\eta : \tilde{X} \rightarrow X$ . Because of Lemma 3.4 (a), we may assume that  $\tilde{X} = E_2$ . Let  $\Gamma_2$  be the group of covering transformations. Then  $|X| = |E_2|/\Gamma_2$ .

Let  $V_2$  be the vertex set of  $E_2$ . We take origin  $(0, 0)$  is the middle point of the line joining  $u_{0,0}$  and  $u_{1,1}$ . Let  $a = u_{1,0} - u_{0,0}$ ,  $b = u_{0,2} - u_{0,0} \in \mathbb{R}^2$ . Then  $H_2 := \langle x \mapsto x + a, y \mapsto y + b \rangle$  is the group of all the translations of  $E_2$ . Under the action of  $H_2$ , vertices form two orbits. Consider the subgroup  $G_2$  of  $\text{Aut}(E_2)$  generated by  $H_2$  and the map  $x \mapsto -x$ . So,

$$G_2 = \{ \alpha : x \mapsto \varepsilon x + ma + nb : \varepsilon = \pm 1, m, n \in \mathbb{Z} \} \cong H_2 \rtimes \mathbb{Z}_2.$$

Clearly,  $G_2$  acts vertex-transitively on  $E_2$ .

*Claim.* If  $K \leq H_2$  then  $K \trianglelefteq G_2$ .

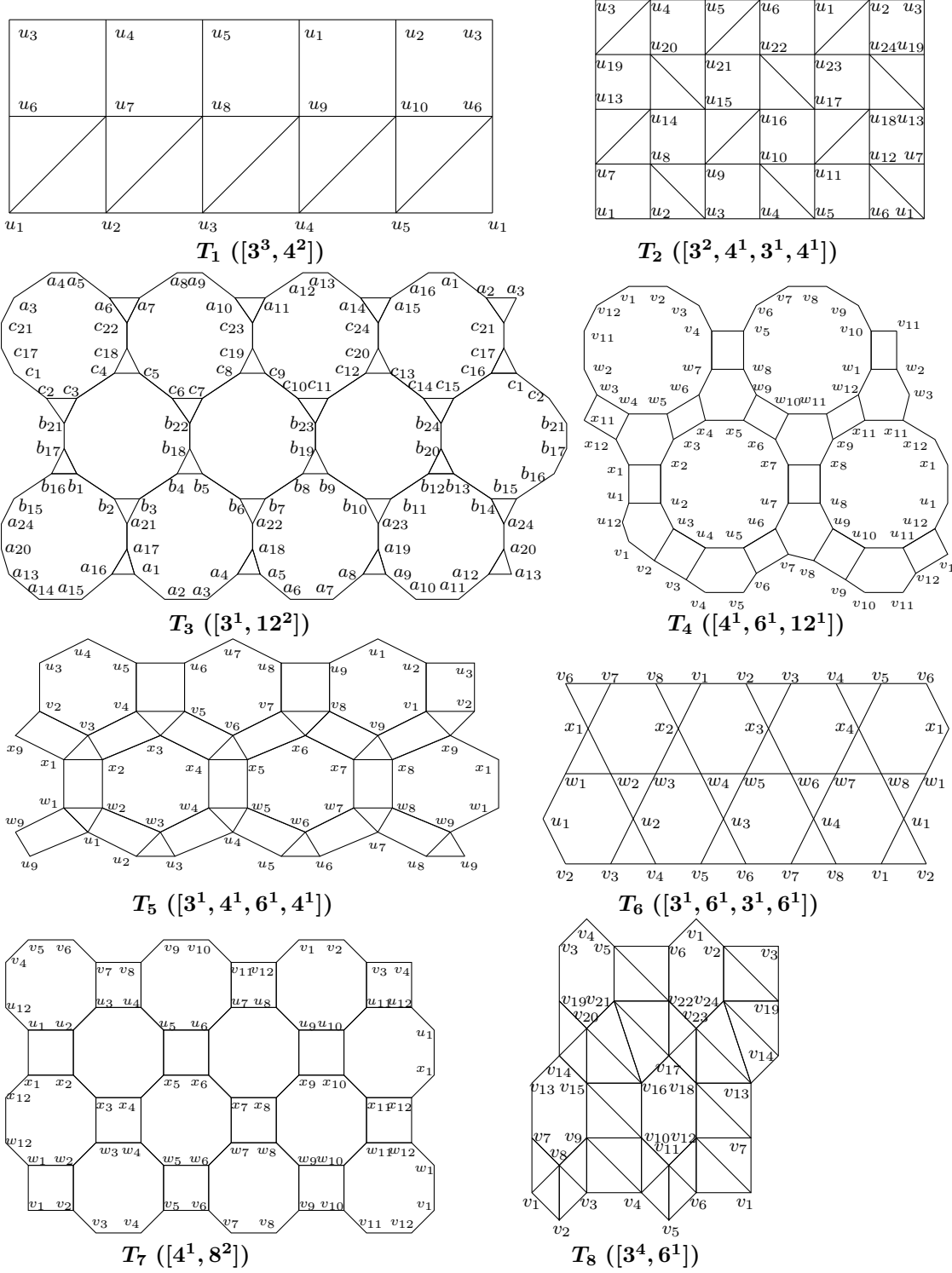
Let  $g \in G_2$  and  $k \in K$ . Assume  $g(x) = \varepsilon x + ma + nb$  and  $k(x) = x + pa + qb$  for some  $m, n, p, q \in \mathbb{Z}$  and  $\varepsilon \in \{1, -1\}$ . Then  $(g \circ k \circ g^{-1})(x) = (g \circ k)(\varepsilon(x - ma - nb)) = g(\varepsilon(x - ma - nb) + pa + qb) = x - ma - nb + \varepsilon(pa + qb) + ma + nb = x + \varepsilon(pa + qb) = k^\varepsilon(x)$ . Thus,  $g \circ k \circ g^{-1} = k^\varepsilon \in K$ . This proves the claim.

For  $\sigma \in \Gamma_2$ ,  $\eta \circ \sigma = \eta$ . So,  $\sigma$  maps a face of the map  $\tilde{X}$  in  $\mathbb{R}^2$  to a face of  $\tilde{X}$  (in  $\mathbb{R}^2$ ). This implies that  $\sigma$  induces an automorphism  $\sigma$  of  $E_2$ . Thus, we can identify  $\Gamma_2$  with a subgroup of  $\text{Aut}(E_2)$ . So,  $X$  is a quotient of  $E_2$  by the subgroup  $\Gamma_2$  of  $\text{Aut}(E_2)$ , where  $\Gamma_2$  has no fixed element (vertex, edge or face). Hence  $\Gamma_2$  consists of translations and glide reflections. Since  $X = E_2/\Gamma_2$  is orientable,  $\Gamma_2$  does not contain any glide reflection. Thus  $\Gamma_2 \leq H_2$ . By the claim,  $\Gamma_2$  is a normal subgroup of  $G_2$ . Since  $G_2$  acts transitively on  $V_2$ ,  $G_2/\Gamma_2$  acts transitively on the vertices of  $E_2/\Gamma_2$ . Thus,  $X$  is vertex-transitive. This completes the proof.  $\square$



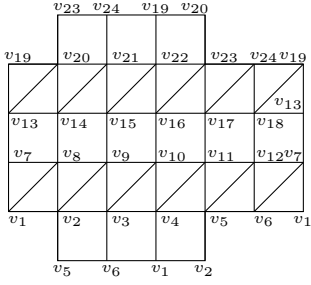
## 4 Examples of maps on the torus and Klein bottle

**Example 4.1.** Eight types of semi-equivelar maps on the torus.

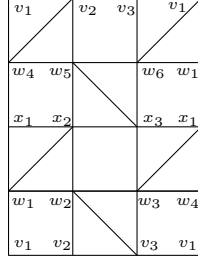


It follows from Theorem 1.6 that the map  $T_1$  is vertex-transitive.

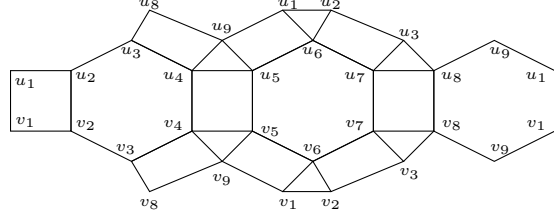
**Example 4.2.** Ten types of semi-equivelar maps on the Klein bottle.



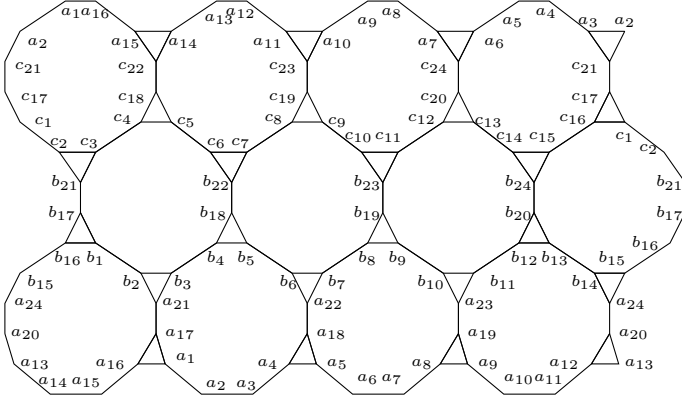
$K_1$  ( $[3^3, 4^2]$ )



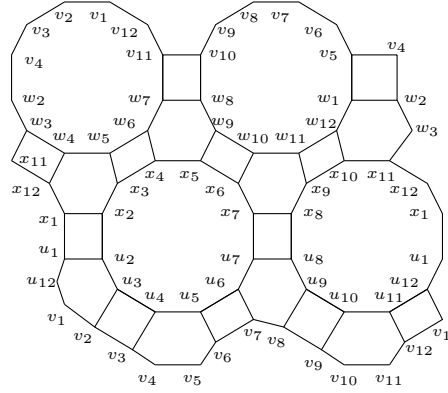
$K_2$  ( $[3^2, 4^1, 3^1, 4^1]$ )



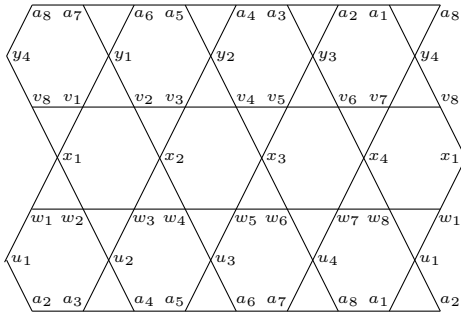
$K_5$  ( $[3^1, 4^1, 6^1, 4^1]$ )



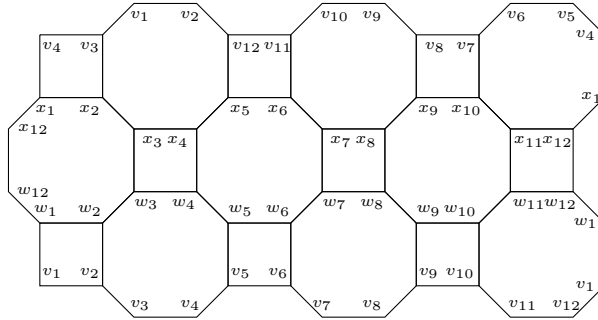
$K_3$  ( $[3, 12^2]$ )



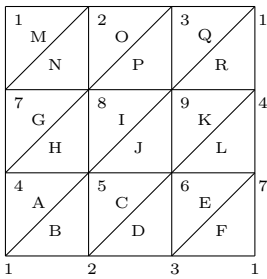
$K_4$  ( $[4^1, 6^1, 12^1]$ )



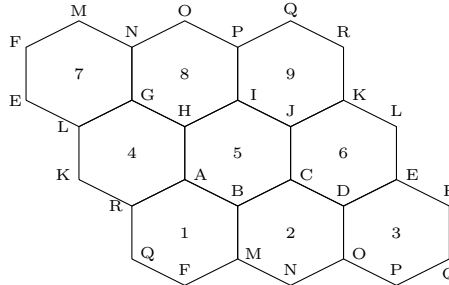
$K_6$  ( $[3^1, 6^1, 3^1, 6^1]$ )



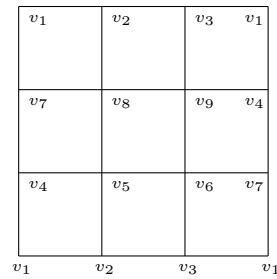
$K_7$  ( $[4^1, 8^2]$ )



$K_8$  ( $[3^6]$ )



$K_9$  ( $[6^3]$ )



$K_{10}$  ( $[4^4]$ )

In the next two proofs, we denote the  $n$ -cycle whose edges are  $u_1u_2, \dots, u_{n-1}u_n, u_nu_1$  by  $C_n(u_1, \dots, u_n)$ . This help us to compare different sizes of cycles.

**Lemma 4.3.** *The semi-equivelar maps  $T_3, \dots, T_8$  in Example 4.1 are not vertex-transitive.*

*Proof.* Let  $G_2$  be the graph whose vertices are the vertices of  $T_2$  and edges are the diagonals of 4-gons of  $T_2$ . Then  $G_2$  is a 2-regular graph. Hence,  $G_2$  is a disjoint union of cycles. Clearly,  $\text{Aut}(T_2)$  acts on  $G_2$ . If the action of  $\text{Aut}(T_2)$  is vertex-transitive on  $T_2$  then it would be vertex-transitive on  $G_2$ . But this is not possible since  $C_4(u_1, u_8, u_{15}, u_{22})$ ,  $C_{12}(u_2, u_7, u_{18}, u_{23}, u_6, u_{11}, u_{16}, u_{21}, u_4, u_9, u_{14}, u_{19})$  are components of  $G_2$  of different sizes.

Let  $G_3$  be the graph whose vertices are the vertices of  $T_3$  and edges are the long diagonals of 12-gons of  $T_3$ . Then  $G_3$  is a 2-regular graph. Hence,  $G_3$  is a disjoint union of cycles. Clearly,  $\text{Aut}(T_3)$  acts on  $G_3$ . If the action of  $\text{Aut}(T_3)$  is vertex-transitive on  $T_3$  then it would be vertex-transitive on  $G_3$ . But this is not possible since  $C_4(a_{17}, a_{22}, a_{19}, a_{24})$  and  $C_{12}(c_1, a_6, b_9, c_{14}, a_1, b_6, c_9, a_{14}, b_1, c_6, a_9, b_{14})$  are components of  $G_3$  of different sizes.

Let  $G_4$  be the graph whose vertices are the vertices of  $T_4$  and edges are the diagonals of 4-gons and long diagonals of 12-gons of  $T_4$ . Then  $G_4$  is a 2-regular graph. Clearly,  $\text{Aut}(T_4)$  acts on  $G_4$ . If the action of  $\text{Aut}(T_4)$  is vertex-transitive on  $T_4$  then it would be vertex-transitive on  $G_4$ . But this is not possible since  $C_4(x_1, u_2, x_7, u_8)$  and  $C_8(v_2, u_4, x_5, w_{10}, v_8, u_{10}, x_{11}, w_4)$  are components of  $G_4$  of different sizes.

Let  $G_5$  be the graph whose vertices are the vertices of  $T_5$  and edges are the diagonals of 4-gons of  $T_5$ . Then  $G_5$  is a 2-regular graph. Hence,  $G_5$  is a disjoint union of cycles. Clearly,  $\text{Aut}(T_5)$  acts on  $G_5$ . If the action of  $\text{Aut}(T_5)$  is vertex-transitive on  $T_5$  then it would be vertex-transitive on  $G_5$ . But this is not possible since  $C_6(x_9, v_3, x_3, v_6, x_6, v_9)$  and  $C_4(u_2, v_2, x_1, w_2)$  are components of  $G_5$  of different sizes.

Let  $G_6$  be the graph whose vertices are the vertices of  $T_6$  and edges are the long diagonals of 6-gons of  $T_6$ . Then  $G_6$  is a 2-regular graph. Hence,  $G_6$  is a disjoint union of cycles. Clearly,  $\text{Aut}(T_6)$  acts on  $G_6$ . If the action of  $\text{Aut}(T_6)$  is vertex-transitive on  $T_6$  then it would be vertex-transitive on  $G_6$ . But this is not possible since  $C_4(u_1, u_2, u_3, u_4)$  and  $C_8(v_2, w_2, v_8, w_8, v_6, w_6, v_4, w_4)$  are components of  $G_6$  of different sizes.

Let  $G_7$  be the graph whose vertices are the vertices of  $T_7$  and edges are the diagonals of 4-gons and common edges between any two 8-gons of  $T_7$ . Then  $G_7$  is a 2-regular graph. Hence,  $G_7$  is a disjoint union of cycles. Clearly,  $\text{Aut}(T_7)$  acts on  $G_7$ . If the action of  $\text{Aut}(T_7)$  is vertex-transitive on  $T_7$  then it would be vertex-transitive on  $G_7$ . But this is not possible since  $C_8(v_1, w_2, w_3, x_4, x_5, u_6, u_7, v_8)$  and  $C_{16}(v_2, w_1, w_{12}, x_{11}, x_{10}, u_9, u_8, v_7, v_6, w_5, w_4, x_3, x_2, u_1, u_{12}, v_3)$  are components of  $G_7$  of different sizes.

We call an edge  $uv$  of  $T_8$  *nice* if at  $u$  (respectively, at  $v$ ) three 3-gons containing  $u$  (respectively,  $v$ ) lie on one side of  $uv$  and one on the other side of  $uv$ . (For example,  $v_{10}v_{15}$  is nice). Observe that there is exactly one nice edge in  $T_8$  through each vertex. Let  $G_8$  be the graph whose vertices are the vertices of  $T_8$  and edges are the nice edges and the long diagonals of 6-gons. Then  $G_8$  is a 2-regular graph. Hence,  $G_8$  is a disjoint union of cycles. Clearly,  $\text{Aut}(T_8)$  acts on  $G_8$ . If the action of  $\text{Aut}(T_8)$  is vertex-transitive on  $T_8$  then it would be vertex-transitive on  $G_8$ . But this is not possible since  $C_4(v_7, v_{15}, v_{10}, v_{18})$  and  $C_8(v_1, v_{23}, v_{17}, v_{11}, v_4, v_{20}, v_{14}, v_8)$  are components of  $G_8$  of different sizes.  $\square$

*Proof of Theorem 1.7.* The result follows from Lemma 4.3.  $\square$

**Lemma 4.4.** *The maps  $K_1, \dots, K_{10}$  are not vertex-transitive.*

*Proof.* Let  $H_1$  be the graph whose vertices are the vertices of  $K_1$  and edges are the diagonals of 4-gons of  $K_1$ . Then  $H_1$  is a 2-regular graph. Hence,  $H_1$  is a disjoint union of cycles. Clearly,  $\text{Aut}(K_1)$  acts on  $H_1$ . If the action of  $\text{Aut}(K_1)$  is vertex-transitive on  $K_1$  then it

would be vertex-transitive on  $H_1$ . But this is not possible since  $C_6(v_7, v_{14}, v_9, v_{16}, v_{11}, v_{18})$  and  $C_3(v_{20}, v_{24}, v_{22})$  are two components of  $H_1$  of different sizes.

There are exactly two induced 3-cycles in  $K_2$ , namely,  $C_3(x_1, x_2, x_3)$  and  $C_3(v_1, v_2, v_3)$ . So, some vertices of  $K_2$  are in an induced 3-cycle and some are not. Therefore, the action of  $\text{Aut}(K_2)$  on  $K_2$  can not be vertex-transitive.

Like  $G_3$  in the proof of Lemma 4.3, let  $H_3$  be the graph whose vertices are the vertices of  $K_3$  and edges are the long diagonals of 12-gons of  $K_3$ . Then,  $\text{Aut}(K_3)$  acts on the 2-regular graph  $H_3$ . If the action of  $\text{Aut}(K_3)$  is vertex-transitive on  $K_3$  then it would be vertex-transitive on  $H_3$ . But this is not possible since  $C_4(a_{17}, a_{22}, a_{19}, a_{24})$  and  $C_{24}(a_3, b_4, c_3, a_1, b_6, c_9, a_7, b_8, c_7, a_{13}, b_2, c_5, a_{11}, b_{12}, c_{11}, a_9, b_{14}, c_1, a_{15}, b_{16}, c_{15}, a_5, b_{10}, c_{13})$  are components of  $H_3$  of different sizes.

Let  $H_4$  be the graph whose vertices are the vertices of  $K_4$  and edges are the diagonals of 4-gons and long diagonals of 12-gons of  $K_4$  (like  $G_4$  in the proof of Lemma 4.3). Then,  $\text{Aut}(K_4)$  acts on the 2-regular graph  $H_4$ . If the action of  $\text{Aut}(K_4)$  is vertex-transitive on  $K_4$  then it would be vertex-transitive on  $H_4$ . But this is not possible since  $C_4(v_5, w_2, v_{11}, w_8)$  and  $C_8(v_2, u_4, x_5, w_{10}, v_7, u_5, x_4, w_5)$  are components of  $H_4$  of different sizes.

Let  $H_5$  be the graph whose vertices are the vertices of  $K_5$  and edges are the diagonals of 4-gons in  $K_5$  (like  $G_5$ ). Then,  $\text{Aut}(K_5)$  acts on the 2-regular graph  $H_5$ . If the action of  $\text{Aut}(K_5)$  is vertex-transitive on  $K_5$  then it would be vertex-transitive on  $H_5$ . But this is not possible since  $C_{12}(v_1, u_2, u_7, v_8, v_4, u_5, u_1, v_2, v_7, u_8, u_4, v_5)$  and  $C_3(u_3, u_9, u_6)$  are components of  $H_5$  of different sizes.

Let  $H_6$  be the graph whose vertices are the vertices of  $K_6$  and edges are the long diagonals of 6-gons of  $K_6$  (like  $G_6$ ). Then,  $\text{Aut}(K_6)$  acts on the 2-regular graph  $H_6$ . If the action of  $\text{Aut}(K_6)$  is vertex-transitive on  $K_6$  then it would be vertex-transitive on  $H_6$ . But this is not possible since  $C_{24}(a_2, w_2, v_2, a_5, w_3, v_1, a_8, w_8, v_8, a_7, w_5, v_3, a_6, w_6, v_6, a_1, w_7, v_5, a_4, w_4, v_4, a_3, w_1, v_7)$  and  $C_4(u_1, u_2, u_3, u_4)$  are components of  $H_6$  of different sizes.

Let  $H_7$  be the graph whose vertices are the vertices of  $K_7$  and edges are the diagonals of 4-gons and common edges between any two 8-gons in  $K_7$  (like  $G_7$ ). Then  $\text{Aut}(K_7)$  acts on the 2-regular graph  $H_7$ . If the action of  $\text{Aut}(K_7)$  is vertex-transitive on  $K_7$  then it would be vertex-transitive on  $H_7$ . But this is not possible since  $C_{24}(v_1, w_2, w_3, x_4, x_5, v_{11}, v_{10}, w_9, w_8, x_7, x_6, v_{12}, v_2, w_1, w_{12}, x_{11}, x_{10}, v_8, v_9, w_{10}, w_{11}, x_{12}, x_1, v_3)$  and  $C_{12}(v_5, w_6, w_7, x_8, x_9, v_7, v_6, w_5, w_4, x_3, x_2, v_4)$  are components of  $H_7$  of different sizes.

Let  $\text{Skel}_1(K_8)$  be the edge graph of  $K_8$  and  $N_8$  be the non-edge graph (i.e., the complement of  $\text{Skel}_1(K_8)$ ) of  $K_8$ . If  $\text{Aut}(K_8)$  acts vertex-transitively then  $\text{Aut}(K_8)$  acts vertex-transitively on  $\text{Skel}_1(K_8)$  and hence on  $N_8$ . But, this is not possible since  $N_8$  is the union of two cycles of different lengths, namely,  $N_8 = C_6(2, 4, 3, 5, 7, 9) \sqcup C_3(1, 6, 8)$ .

Consider the triangles  $C = 256$  and  $O = 238$  in  $K_8$ . If there exists  $\alpha \in \text{Aut}(K_8)$  such that  $\alpha(C) = O$  then  $\alpha$  acts on  $N_8 = C_6(2, 4, 3, 5, 7, 9) \sqcup C_3(1, 6, 8)$  and hence  $\alpha(6) = 8$ ,  $\alpha(\{2, 5\}) = \{2, 3\}$ . This is not possible, since 25 is a long diagonal in  $C_6(2, 4, 3, 5, 7, 9)$  where as 23 is a short diagonal in  $C_6(2, 4, 3, 5, 7, 9)$ . Thus, the action of  $\text{Aut}(K_8)$  is not face-transitive on  $K_8$ . Observe that  $K_9$  is the dual of  $K_8$ . Hence  $\text{Aut}(K_9) = \text{Aut}(K_8)$  is not vertex-transitive on  $K_9$ .

There are exactly four induced 3-cycles in  $K_{10}$ , namely,  $C_3(v_1, v_2, v_3)$ ,  $C_3(v_1, v_4, v_7)$ ,  $C_3(v_2, v_5, v_8)$  and  $C_3(v_3, v_6, v_7)$ . Let  $H_{10} := C_3(v_1, v_2, v_3) \cup C_3(v_1, v_4, v_7) \cup C_3(v_2, v_5, v_8) \cup C_3(v_3, v_6, v_7)$ . Clearly,  $\text{Aut}(K_{10})$  acts on  $H_{10}$ . If the action of  $\text{Aut}(K_{10})$  is vertex-transitive on  $K_{10}$  then it would be vertex-transitive on  $H_{10}$ . But this is not possible since the degrees of all the vertices in  $H_{10}$  are not same.  $\square$

*Proof of Theorem 1.8.* The result follows from Lemma 4.4. □

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